Identity Matrix

Question: Given that Matrices **A, B** and **X** are of order *m by m*, and an identity matrix of *m by m* dimension is given by $\mathbf{I_m}$. Solve for Matrix **X** using the following equation.

$$\boldsymbol{AX - BA = I_m}$$

Concept:

- An identity matrix is a square matrix that looks like the following:

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \text{ Or } \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \text{ Or } \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

  Or in general, it can be written as:

$$\boldsymbol{Identity\ Matrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix};$$

- We can get an Identity Matrix by multiplying a **square matrix** by its **inverse**, if and only if, this **square matrix** can be "inversed" (or invertible or non-singular) in the first place.

  Thus, if there exist a square matrix called **S**, and there exist an inverse of **S** (often written as $\mathbf{S}^{-1}$), then $\boldsymbol{SS^{-1}} = \boldsymbol{Identity\ Matrix}$.

Solution:

Since all the matrices have an order of *m by* m, let:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a_{1,1}} & \cdots & \boldsymbol{a_{1,m}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{a_{m,1}} & \cdots & \boldsymbol{a_{m,m}} \end{bmatrix};\ \boldsymbol{B} = \begin{bmatrix} \boldsymbol{b_{1,1}} & \cdots & \boldsymbol{b_{1,m}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{b_{m,1}} & \cdots & \boldsymbol{b_{m,m}} \end{bmatrix};$$

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x_{1,1}} & \cdots & \boldsymbol{x_{1,m}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{x_{m,1}} & \cdots & \boldsymbol{x_{m,m}} \end{bmatrix};\ \boldsymbol{I_m} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix};$$

$\boldsymbol{AX - BA = I_m}$ is rewritten in the form $[\mathbf{A}][\mathbf{X}] - [\mathbf{B}][\mathbf{A}] = [\mathbf{I_m}]$

$$[\mathbf{A}][\mathbf{X}] - [\mathbf{B}][\mathbf{A}] = [\mathbf{I_m}]$$

$$[\mathbf{A}][\mathbf{X}] = [\mathbf{I_m}] + [\mathbf{B}][\mathbf{A}]$$

We need to remove [A] from the left side of the equation. Unlike in algebra where we can divide a variable, we cannot divide a matrix. Thus we have to make use the inverse property of a matrix.

$$\underbrace{{\color{red}[\mathbf{A}]^{-1}}[\mathbf{A}]}[\mathbf{X}] = {\color{red}[\mathbf{A}]^{-1}}([\mathbf{I_m}] + [\mathbf{B}][\mathbf{A}])$$

Recall that multiplying a square matrix by its inverse will result in an identity matrix.

$${\color{green}[\boldsymbol{I_m}]}[\mathbf{X}] = {\color{red}[\mathbf{A}]^{-1}}([\mathbf{I_m}] + [\mathbf{B}][\mathbf{A}])$$

${\color{green}\boldsymbol{I_m}}$ "disappeared" as any matrix multiplied by an identity matrix will yield the original matrix.

$$[\mathbf{X}] = {\color{red}[\mathbf{A}]^{-1}}([\mathbf{I_m}] + [\mathbf{B}][\mathbf{A}])$$